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FREQUENCY RESPONSE FUNCTION ESTIMATION VIA A ROBUST WAVELET DE-NOISING METHOD

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The FRF (frequency response function) estimation can be performed by the vibration analysis of a linear time-invariant dynamic system. Since a single FRF estimate is highly sensitive to measurement errors of input/output signals, the mean averaging of repeatedly observed FRF estimates is employed in most of the practical applications. The main result of this work is in reducing the number of averaging operations and enhancing estimation accuracy by using a robust wavelet de-noising method. This approach removes outliers and zero-mean Gaussian noise simultaneously and effectively while preserving most of the important signal features of a true FRF with a dramatically smaller number of operations as compared with the traditional mean-averaging procedure. The robust wavelet de-noising method is based on a wavelet-related median filtering and a wavelet shrinkage to reduce the effect of outliers and zero-mean Gaussian noise respectively. The effectiveness of the present FRF estimation technique is demonstrated using both simulated and experimental data.

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1. INTRODUCTION

The FRF estimation of a linear time-invariant dynamic system is common for system identification, and various methods including traditional Fourier-based filtering methods have been proposed to better FRF estimation. In random vibration analysis [1, 2], an FRF estimate obtained from a single observation of input/output signals exhibits a severe distortion due to inevitable measurement errors. Thus, a mean-averaging procedure among noisy FRF estimates may be utilized, and is, in fact, the simplest and the most widely used method in FRF estimation. In the case of some complicated structures, however, very intensive averaging is required to obtain an FRF estimate with an acceptable precision.

The noise appearing in an FRF estimate may be classified in two types; one has a relatively low intensity but contaminates the true signal in most of the time, and the other has a relatively high intensity but exists only in a short period of time. The first type of noise is well characterized by a Gaussian noise model. In this work we call the second type of

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noise *outliers* in a sense that it is neither part of a true signal nor small measurement errors that can be reasonably modelled by a Gaussian noise.

The purpose of this work is in reducing the number of averaging in FRF estimation. To do so, we can apply first a noise-removal algorithm to each FRF estimate and then take the mean average of noise-removed ones. Most of the noise-removal algorithms, however, do not simultaneously remove well a Gaussian noise and outliers. For example, the Fourier-based filtering does not work well on outlier removal; the energy on an outlier is spread over a wide frequency range so that usual low-pass filtering cannot remove outliers efficiently. On the other hand, a median filter is well known to be good at removing outliers. A considerably long mask is, however, required for the median filter to suppress a Gaussian noise effectively, which results in an over-smoothed FRF estimate.

Recently, there has been a great interest in the use of wavelet bases in signal/image processing [3, 4], statistical applications [5, 6], etc. One of the successful wavelet applications is the recovery of a signal corrupted by a Gaussian noise. For a Gaussian noise removal, Donoho and Johnstone developed a *wavelet shrinkage* method [see, e.g., references [5–7]), which either shrinks all wavelet coefficients towards zero by a certain amount (*soft thresholding*) or sets to zeros the wavelet coefficients that are less than a certain value (*hard thresholding*). This shrinkage method has been proven to give a better performance in Gaussian noise reduction than classical Fourier-based methods. Unfortunately, the wavelet shrinkage method is not suitable for the removal of outliers which usually appear in an FRF estimate. Recently, a wavelet-based de-noising method for the FRF estimation was suggested by Bodin and Wahlberg [8, 9]. They combined the wavelet shrinkage method with a preconditioning step operated by a Hanning window for FRF de-noising. Their approach, however, has no consideration of a noise in an input signal.

In this paper, we employ a robust wavelet de-noising method in order to reduce dramatically the number of averaging operations and improve the quality of FRF estimates. This method is based on a robust wavelet decomposition, which is introduced in reference [10]. Unlike the approach by Bodin and Wahlberg [8, 9], the present FRF estimation method can pin-point outliers and a Gaussian noise by employing the robust wavelet de-noising scheme consisting of repeated applications of wavelet-based median filtering and wavelet shrinkage.

We have implemented this method and present several simulated and experimental results. These results show that outliers and zero-mean Gaussian noise in an FRF can be simultaneously and effectively removed while most of the important signal features of a true FRF are preserved. Furthermore, the number of averaging operations is dramatically smaller as compared with the traditional procedure where only the straightforward mean-averaging technique is employed.

This paper is organized as follows. Section 2 briefly reviews the basic wavelet theory and a wavelet de-noising method for a Gaussian noise. Section 3 explains the common procedure of the FRF estimation and the difficulty in dealing with the noise in an FRF estimate. Section 4 presents new robust wavelet decompositions, which are resistant to outliers, and a corresponding de-noising method for the FRF estimation. Section 5 investigates the performance of the proposed method for numerical and experimental data. Finally, section 6 concludes this work.

2. WAVELETS AND GAUSSIAN NOISE REMOVAL

In this section, we briefly review the basic wavelet theory and the wavelet-related de-noising method suggested by Donoho and Johnstone. More details can be found in references [6, 11].

The term (orthogonal) "wavelet" itself refers to a real-valued function ψ, defined on the whole real line $\mathbb{R}$, with the combination of integer shift and dyadic dilation, that is, $\psi_{k,j} = 2^{k/2}\psi(2^k \cdot - j)$, $k, j \in \mathbb{Z}$, yields an orthonormal basis of the set of all energy bounded (real-valued) signals, $L^2(\mathbb{R})$. Thus, any signal $f \in L^2(\mathbb{R})$ can be represented as a series

$$f = \sum_{k \in \mathbb{R}} \sum_{j \in \mathbb{R}} D_k[j]\psi_{k,j}, \quad D_k[j] = \langle f, \psi_{k,j} \rangle, \tag{1}$$

where the inner product of f and g in $L^2(\mathbb{R})$ is defined by $\langle f, g \rangle = \int f(x)g(x)\,\mathrm{d}x$.

The scaling function ϕ is associated with ψ, and from which one generates functions $\phi_{k,j} = 2^{k/2}\,\phi(2^{k\cdot} - j)$, $k, j \in \mathbb{Z}$. With $C_k[j] = \langle f, \phi_{k,j} \rangle$, we have

$$f = \sum_{k \geqslant k_0} \sum_{j \in \mathbb{Z}} D_k[j]\psi_{k,j} + \sum_{l \in \mathbb{Z}} C_{k_0}[l]\phi_{k_0,l} \tag{2}$$

for each integer k_0. For a given scaling function ϕ, we assume that there exists a finite sequence (h_n) such that $\phi = \sum_n h_n \phi_{1,n}$ and $\psi = \sum_n g_n \phi_{1,n}$, where $g_n = (-1)^n h_{-n+1}$. These equations prove [for details, see, e.g., reference [11]] that

$$C_k[j] = \sum_n h_{n-2j} C_{k+1}[n], \qquad D_k[j] = \sum_n g_{n-2j} C_{k+1}[n], \tag{3}$$

and

$$C_{k+1}[j] = \sum_n h_{j-2n} C_k[n] + g_{j-2n} D_k[n]. \tag{4}$$

We call the first and second equations in equation (3) the low-pass fast wavelet transform (FWT) and high-pass FWT respectively.

When one is concerned with a discrete signal (f_n), $n = 0, 1, \ldots, 2^m - 1$, (in this work, for the simplicity of the presentation, we shall consider only a discrete signal whose length is 2^m for some positive integer m), a common practice in wavelet-based signal processing assumes that there exists a signal f such that $f_n = 2^{m/2}\langle f, \phi_{m,n} \rangle$, i.e., $f_n = 2^{m/2}\, C_m[n]$. For instance, if ϕ is the characteristic function of [0, 1], i.e., $\phi(x) = 1$ for $x \in [0, 1]$, and $\phi(x) = 0$ for $x \notin [0, 1]$, the sample value f_n is the average value of the continuous signal f over the interval $[n2^{-m}, (n+1)2^{-m}]$. With this observation model, we can compute $C_{k_0}, D_{k_0}, D_{k_0+1}, \ldots, D_{m-1}$ from (f_n) by using equation (3) successively. We call this procedure fast wavelet transform (FWT). Inverse fast wavelet transform (IFWT) is the procedure of obtaining (f_n) from C_{k_0}, $D_{k_0}, D_{k_0+1}, \ldots, D_{m-1}$ by using equation (4) successively.

One of the major advantages of FWT is the so-called "energy concentration property", which means that most of the energy in the signal is concentrated on a few wavelet coefficients. On the other hand, the Gaussian noise is spread to all wavelet coefficients with the same variance. Using this property, Donoho and Johnstone [6] suggested a wavelet-related de-noising method. Suppose a noisy data $Y = f + Z$, where f is a true signal contaminated by a Gaussian noise $Z \sim \mathcal{N}(0, \sigma^2)$. The de-noising method suggested by Donoho and Johnstone employs three steps:

- Apply FWT to a set of noisy data Y to get its wavelet coefficients.
- Use soft or hard thresholding to the wavelet coefficients in order to reduce a Gaussian noise.
- Apply IFWT to the resulting coefficients from the previous step.

The definitions of soft threshold $S_\mu(x)$ and hard thresholding $H_\mu(x)$ are

$$S_\mu(x) = \begin{cases} x - \mu, & x > \mu, \\ 0, & |x| \leqslant \mu, \\ x + \mu, & x < -\mu, \end{cases}$$

and

$$H_\mu(x) = \begin{cases} x, & |x| > \mu, \\ 0, & |x| \leqslant \mu. \end{cases}$$

This method provides asymptotically near-optimal results for Gaussian noise removal if a well-chosen thresholding value μ is employed, and gives better Gaussian noise reduction than classical Fourier-based methods. A near-optimal threshold value μ can be obtained by several ways [3, 5, 6]. We state a method for future use:

$$\mu_{m-1} = \sqrt{2 \log 2^m}\sigma, \qquad \mu_k = \mu_{k+1}/2, \tag{5}$$

where σ denotes the standard deviation of a Gaussian noise.

3. FRF ESTIMATION

In this section, we shall briefly review the common procedure of FRF estimation from noisy data and some difficulties in doing so.

The linear time-invariant dynamic system can be formulated as

$$y(t) = h * u(t) + v(t), \quad v(t) \sim \mathcal{N}(0, \sigma_v^2), \tag{6}$$

where $h * u(t) = \int_{-\infty}^{t} h(t-\tau)u(\tau)\,\mathrm{d}\tau$. In this model we assume that $u(t)$ is a true input signal and $y(t)$ is a measured output signal corrupted by a Gaussian noise $v(t)$. We also assume that the measurement error in the true input signal $u(t)$ follows a zero-mean Gaussian noise model. Thus, the measured input signal $x(t)$ is modelled by

$$x(t) = u(t) + n(t), \quad n(t) \sim \mathcal{N}(0, \sigma_n^2), \tag{7}$$

The FRF $H(\omega)$ of the system is defined by

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t. \tag{8}$$

The goal is to obtain an accurate estimate estimate of the true FRF $H(\omega)$ from $x(t)$ and $y(t)$.

In a stationary random process, the linear relation of input and output is commonly given by cross- and auto-spectral density functions $S_{xy}(\omega)$ and $S_{xx}(\omega)$ [for definitions, see, e.g., references [1, 2]] respectively. Using a window function to reduce a leakage error, an estimated FRF $\tilde{H}(\omega)$ can be given by

$$\tilde{H}(\omega) = \frac{W(\omega) * S_{xy}(\omega)}{W(\omega) * S_{xx}(\omega)} \tag{9}$$

with a complex-valued window function $W(\omega)$. Thus, an estimated FRF $\tilde{H}(\omega)$ can be expressed as

$$\tilde{H}(\omega) = H(\omega) + Z(\omega), \tag{10}$$

where $H(\omega)$ is the true FRF and $Z(\omega)$ is a complex-valued noise generated by measurement errors in input/output signals and the use of the window function. Hence $Z(\omega)$ depends on

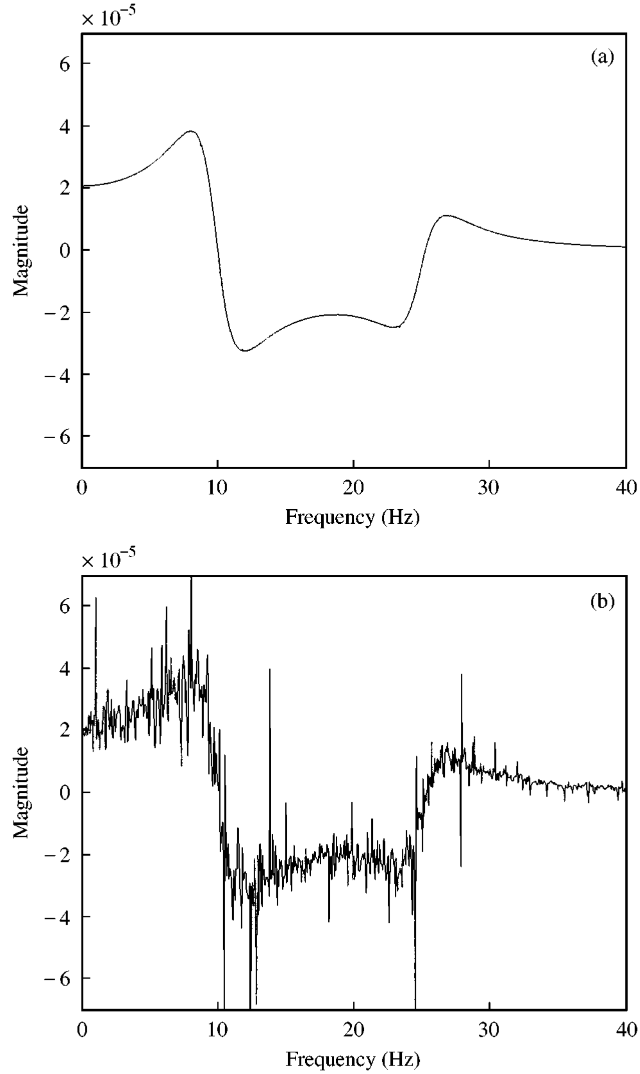

Figure 1. True and noisy FRF (real part). (a) Real $(H(\omega))$; (b) real $(\tilde{H}(\omega))$.

h, u, n, v and W. This implies that $Z(\omega)$ no longer follows a Gaussian noise model even if input and output signals are contaminated with pure Gaussian noise models.

Figure 1(a) shows the real part of the true FRF of a two-degrees-of-freedom (2-d.o.f.) linear system. To demonstrate an effect of measurement errors in input/output signals, a Gaussian noise $n(t) \sim \mathcal{N}(0,\ 500^2)$ is added to a true input signal $u(t)$ and another Gaussian noise $v(t) \sim \mathcal{N}(0, 0{\cdot}002^2)$ to a true output signal $h * u(t)$. Figure 1(b) shows the real part of the estimated FRF from a single set of input/output signals, with the Hanning window function in equation (9). As we can see from Figure 1(b), the FRF estimate from a single measurement suffers from a severe noise.

By comparing Figures 1(a) and (b), one can immediately see that the noise $Z(\omega)$ in equation (10) is very far from a Gaussian noise. In this paper, we assume the following noise model:

$$Z(\omega) = G(\omega) + O(\omega), \tag{11}$$

where $G(\omega)$ is a Gaussian noise and $O(\omega)$ is a set of outliers, which are neither part of a true signal nor small measurement errors that can be reasonably modelled by a Gaussian noise.

Let

$$\hat{H}^{[M]}(\omega) = \frac{1}{M} \sum_{i=1}^{M} \bar{H}_i(\omega), \tag{12}$$

where $\tilde{H}_i$ is the estimated FRF from the ith input/output signals by equation (9). If we increase M, the number of measurements, then the estimated FRF $\hat{H}^{[M]}$ in equation (12) would eventually converge to the true FRF H. The main difficulty of the FRF estimation by equation (12), however, is slow convergence. That is, to get a satisfactory FRF estimate in equation (12), one needs to endure with a time-consuming measurement procedure.

The slow rate of convergence in $\hat{H}^{[M]}$ to H is largely due to a Gaussian noise and wide-spread outliers in $\tilde{H}_i$. To reduce the number of operations in equation (12) one can apply a noise removal algorithm to $\tilde{H}_i$ to get a noise-reduced estimate $\tilde{H}_i^{\#}$ and take the mean among $\tilde{H}_i^{\#}$ instead of $\tilde{H}_i$ in equation (12). Most of the noise-removal algorithms, however, do not simultaneously remove well a Gaussian noise and outliers. For example, the standard wavelet de-noising method does not work well on outlier removal. Notice that an outlier exists in a very short period of time and has relatively higher energy. Thus, most of the energy on an outlier is concentrated on a few wavelet coefficients $D_k(j)$ that are too large to be removed by the standard wavelet de-noising method without a degradation of signal. The Fourier-based filtering makes the situation worse; the energy in an outlier is spread over a wide frequency range, and thus it is not easily removed by the usual low-pass filtering. Moreover, the Fourier-based filtering hardly distinguishes the outlier from a high-pitched signal. On the other hand, the median filter is known to be good at removing outliers. But, a considerably long mask is required for the median filter to suppress the Gaussian noise effectively, which results in an over-smoothed FRF estimate. To overcome the described weakness in the standard wavelet de-noising method, the Fourier-based filtering and the median filtering, we suggest the use of a robust wavelet de-noising method. This is the topic of the next section.

4. ROBUST WAVELET DE-NOISING

In this section, we shall explain a robust wavelet decomposition proposed by Bruce *et al.* [10] with some changes in notations and terminology. We shall also present how to apply this robust wavelet decomposition to the FRF estimation.

We begin with the outlier-resistant wavelet decomposition proposed by Bruce *et al.* [10]. Let $C_m^{\#} = C_m$, where C_m is the sequence of the finest wavelet coefficients. Starting from coefficients $C_m^{\#}$, for each level $k\,(k \leqslant m)$ the sequence of coefficients $C_k^{\#}$ is decomposed into three components:

- Outlier coefficients O_k given by $O_k[j] = S_\varepsilon(C_k^{\#}[j] - \tilde{C}_k^{\#}[j])$, where $S_\varepsilon(x)$ is the soft thresholding and $\tilde{C}_k^{\#}$ is a median-filtered sequence of $C_k^{\#}$.

- New smooth wavelet coefficients $C_{k-1}^{\#}$ obtained by applying the low-pass FWT to the outlier-free coefficients $C_k^{\#} - O_k$.

- New detail wavelet coefficients $D_{k-1}^{\#}$ obtained by applying the high-pass FWT to the outlier-free coefficients $C_k^{\#} - O_k$.

The length of the median filter is determined to avoid outliers leaking into coarse levels. In practice, using a median filter of length 5 or 7 is usually sufficient. Also, the depth, at

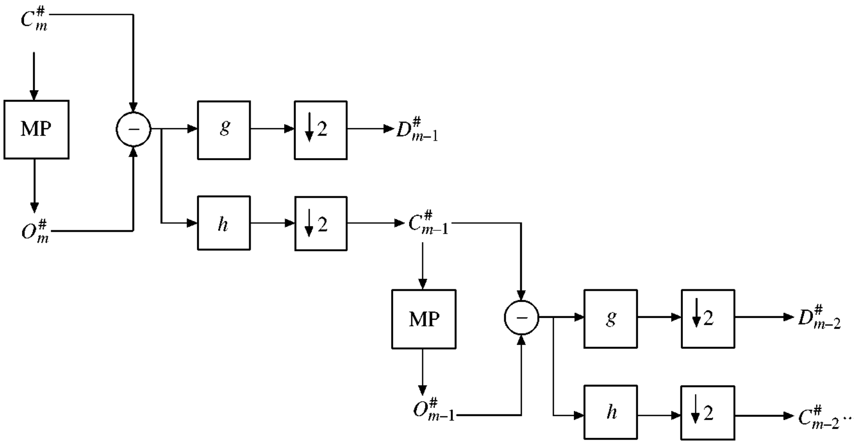

Figure 2. Robust wavelet decomposition.

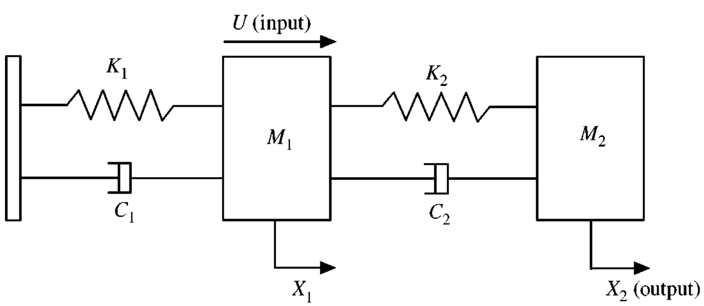

Figure 3. 2-d.o.f. simulation model.

which median filters are operated, is determined selectively to localize the smoothing effect of median filters.

Figure 2 depicts the systematic procedure of robust wavelet decomposition, where "MP" indicates the procedure $O_k[j] = S_\varepsilon(C_k^{\#}[j] - \tilde{C}_k^{\#}[j])$. In our applications, the median filtering was used for the first two levels and the standard wavelet decomposition and the corresponding shrinkage method was used for the remaining levels. This method behaves like the standard wavelet shrinkage for removal of a Gaussian noise, but in contrast with the standard wavelet shrinkage, it reduces outliers by an additional median filtering effectively. Finally, a noise-removed signal is reconstructed by applying IFWT on the resulting coefficients.

The proposed method requires shrinkage steps in $O_k[j] = S_\varepsilon(C_k^{\#}[j] - \tilde{C}_k^{\#}[j])$ to pinpoint outliers and in $S_{\mu_k}(D_k^{*}[j])$ to remove a Gaussian noise, where D_k^{*} is either $D_k^{\#}$ by the robust wavelet decomposition or D_k by the standard wavelet decomposition. We suggest the use of the threshold value μ_k in equation (5), i.e., $\mu_{m-1} = \sqrt{2 \log 2^m}\sigma$ and $\mu_k = \mu_{k+1}/2$, where σ is the standard deviation of the Gaussian noise G. The standard deviation σ of the Gaussian noise G can be accurately estimated by many statistical methods [12] as long as outliers are not too heavily spread. Recall that the parameter ε in

TABLE 1

System parameters

Mode	Frequency (Hz)	Damping ratio (%)
1	10·0	19·1
2	25·0	7·5

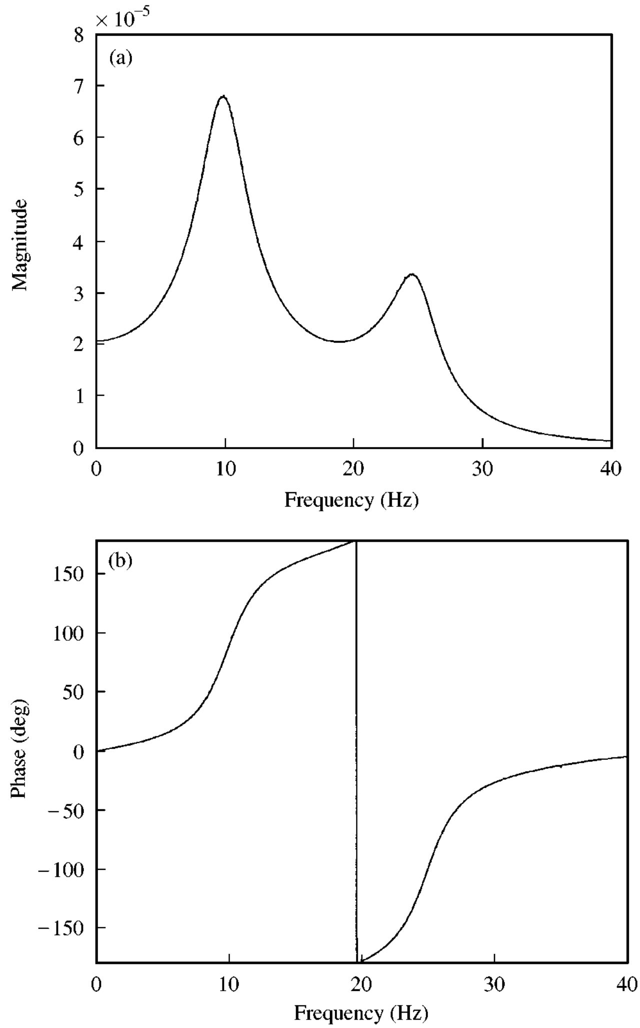

Figure 4. True FRF. (a) Magnitude; (b) phase.

$O_k[j] = S_\varepsilon(C_k^{\#}[j] - \tilde{C}_k^{\#}[j])$ is designed to distinguish outliers from a Gaussian noise. We suggest use of $\varepsilon = 1{\cdot}96\sigma$ that is sufficiently large to have 95% confidence and sufficiently small to locate most of the outliers.

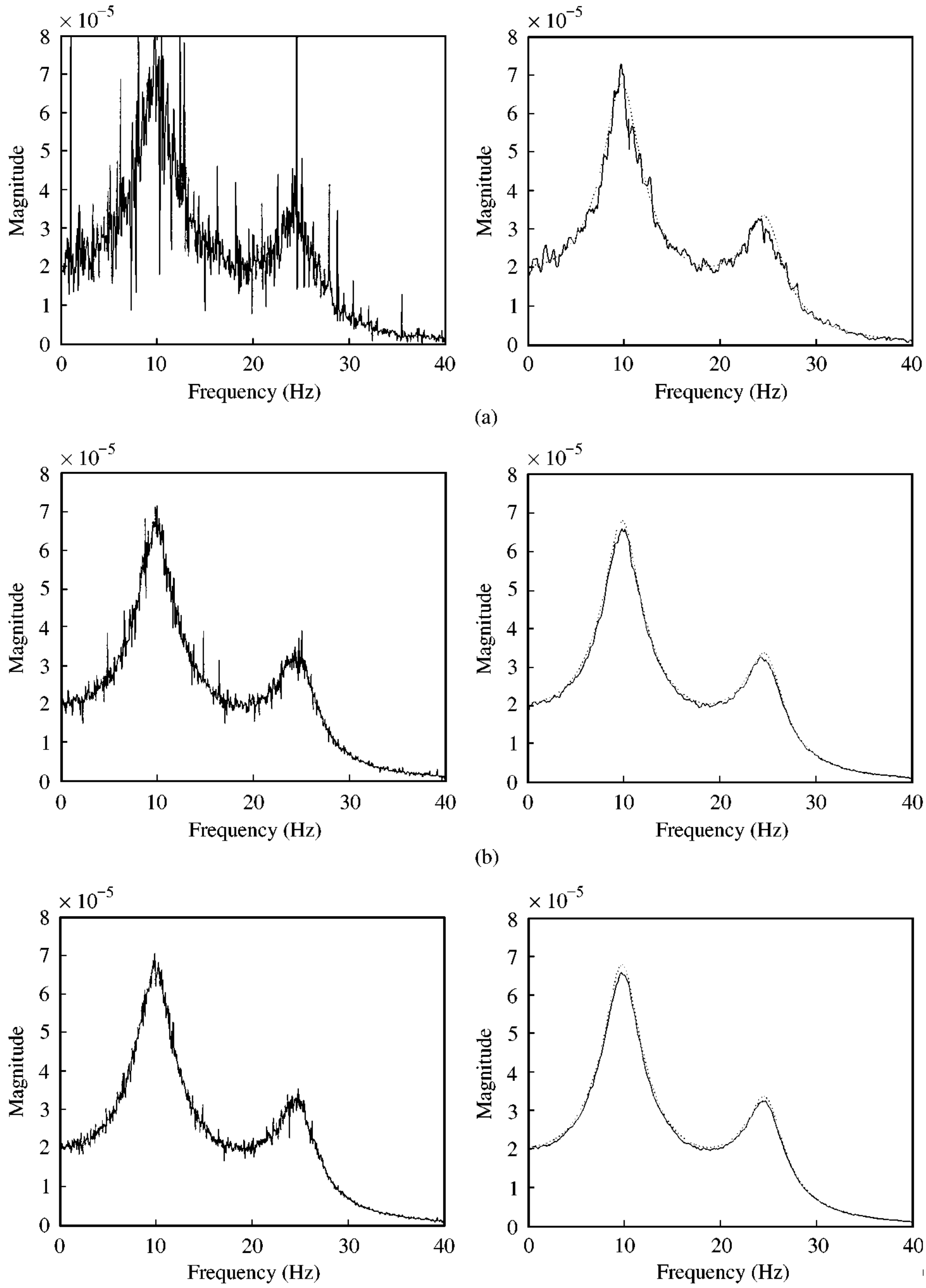

Figure 5. The estimation of the FRF magnitude by the simple mean-averaging (left) and the proposed method (right): (a) single FRF; (b) no. of averages = 20; (c) no. of averages = 50: ·····, true FRF; ——, estimate.

5. APPLICATIONS

We apply the robust wavelet de-noising method to the real and imaginary parts of each FRF estimate $\tilde{H}_i$ separately, and then take the average $\hat{H}^{[M]}$ among the resulting

M-noise-removed FRF's, i.e.,

$$\hat{H}^{[M]}(\omega) = \frac{1}{M} \sum_{i=1}^{M} \tilde{H}_i^{\#}(\omega), \tag{13}$$

where $\tilde{H}_i^{\#}$ is obtained from the estimated FRF $\tilde{H}_i$ [see equation (12)] through the robust wavelet de-noising method.

The method presented in the previous section was applied for the FRF estimation of a 2-d.o.f. linear system ($M_1 = 2$, $M_2 = 0.2$, $C_1 = 40$, $C_2 = 5$, $K_1 = 48567$, $K_2 = 805$) shown in Figure 3. Table 1 gives the system parameters and Figure 4 shows the true FRF.

We generated a random sequence $(u_j)_{j=0,1,\ldots,2047}$ as discrete samples of the true input signal $u(t)$. To simulate a measurement error, we added an independent and identically distributed Gaussian noise with standard deviations $\sigma_n = 500$ ($\mathrm{SNR}_n = 21.25$) and $\sigma_v = 0.002$ ($\mathrm{SNR}_v = 36.09$) to the input signal and the output signal respectively. Here SNR denotes *signal-to-noise ratio* defined by

$$SNR = 10 \log_{10} \frac{\sum_{k=0}^{N-1} |f_k|^2}{N\sigma^2},$$

where (f_k) is a certain original signal, σ is the standard deviation of the added noise and N is the signal length. Again, we used the Hanning function as the window function in equation (9).

To localize a smoothing effect by the median filter, we chose a median filter of length 7 and applied the median filtering to the first two levels. We used the biorthogonal wavelet $\psi_{6,8}$ [11] for this simulation. The threshold values were chosen as described in the previous section, and the wavelet shrinkage was performed up to level 5. We also employed the translation invariant wavelet shrinkage techniques to reduce some artifacts. For details, see reference [13].

Figures 5 and 6 show the performance of the simple mean-averaging and the proposed method in FRF estimation respectively. The performance comparison can be measured by the relative root mean square error (RMSE) ratio,

$$RMSE\ ratio = \frac{\| \hat{H}^{[M]}(\omega) - H(\omega) \|}{\| H(\omega) \|}, \tag{14}$$

where $\hat{H}^{[M]}$ is the estimated FRF by either the mean averaging only or the proposed method. As shown in Figure 7, the proposed method produces smaller estimation errors and has better noise reduction as compared with the simple mean-averaging method.

The proposed method was applied to the FRF estimation of an experimental model shown in Figure 8. Although no exact FRF is known, this example shows that the proposed method is practically useful. Figures 9 and 10 compare the estimated FRF's by simple mean-averaging with those by the proposed method. The advantage of the proposed method over the simple mean-averaging method is clearly shown in these figures.

6. CONCLUSIONS

In this paper, a new FRF estimation technique via a robust wavelet de-noising method combined with the mean averaging was introduced. The robust wavelet denoising method is

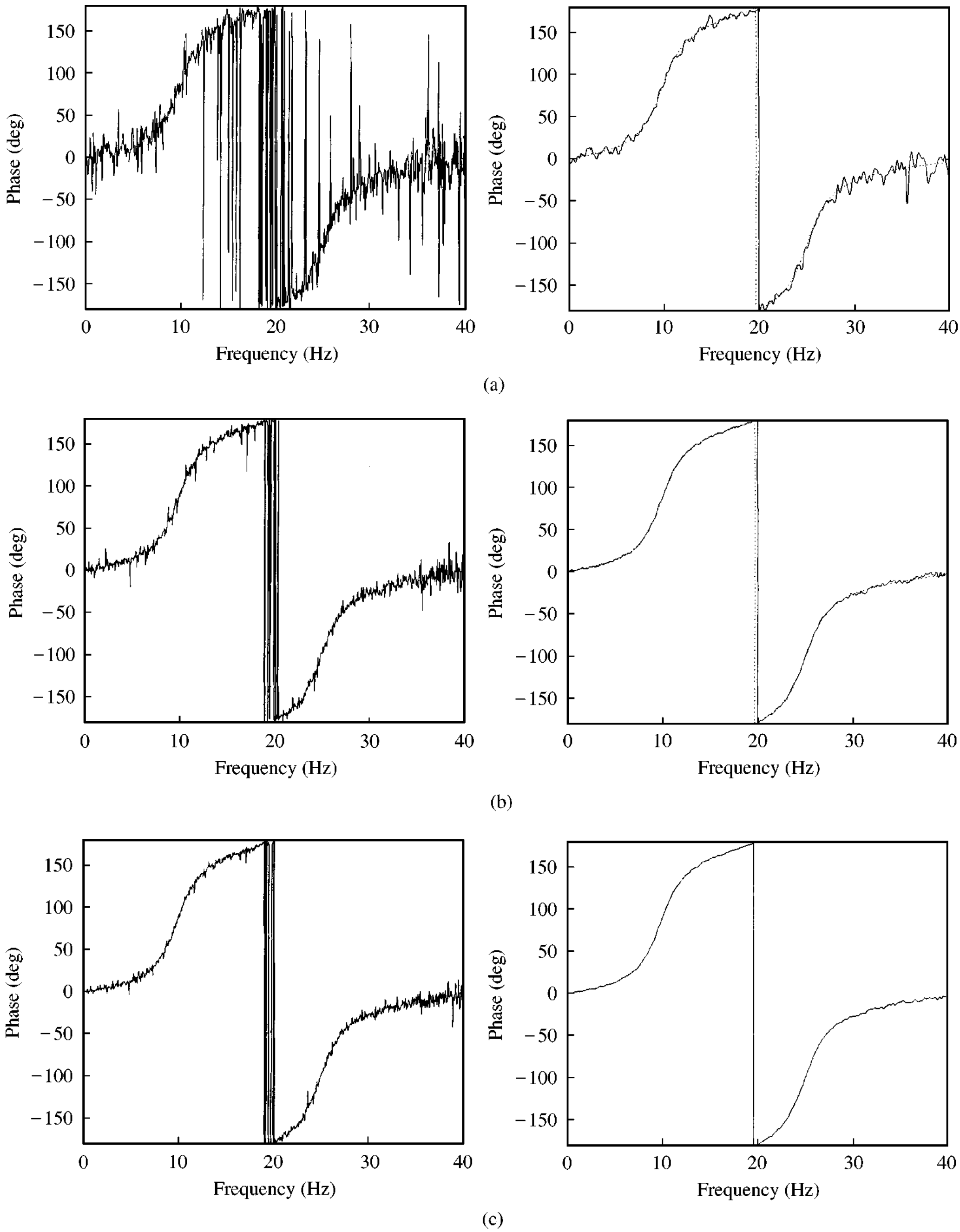

Figure 6. The estimation of the FRF phase by the simple mean-averaging (left) and the proposed method (right): (a) single FRF; (b) no. of averages = 20; (c) no. of averages = 50:·····, true FRF; ——, estimate.

adapted as an alternative tool to overcome the limitation of the standard wavelet shrinkage, Fourier-based filtering and median filtering in removing the Gaussian noise and outliers simultaneously. This method uses a wavelet-related median filtering to suppress outliers

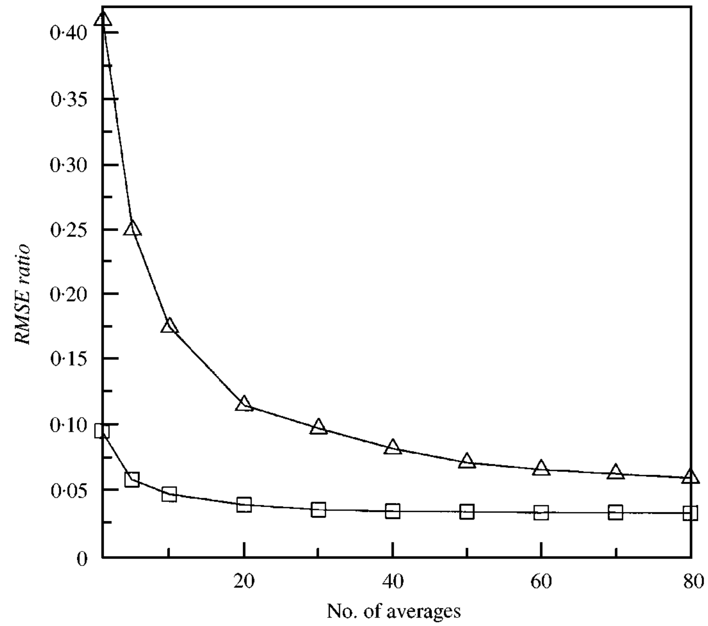

Figure 7. Comparison of the relative *RMSE ratio*: —Δ—, with mean averaging; —□—, with proposed method.

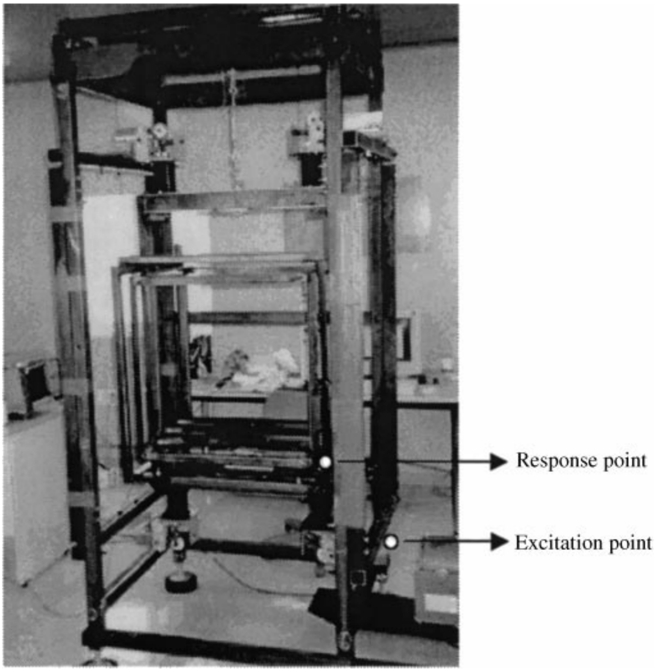

Figure 8. Photo of the experimental model.

while dramatically reducing a Gaussian noise with the wavelet shrinkage. The order of the computational complexity of the proposed method with the translation invariant wavelet shrinkage technique is $\mathcal{O}(N \log N)$, which is equal to that of the Fourier-based method.

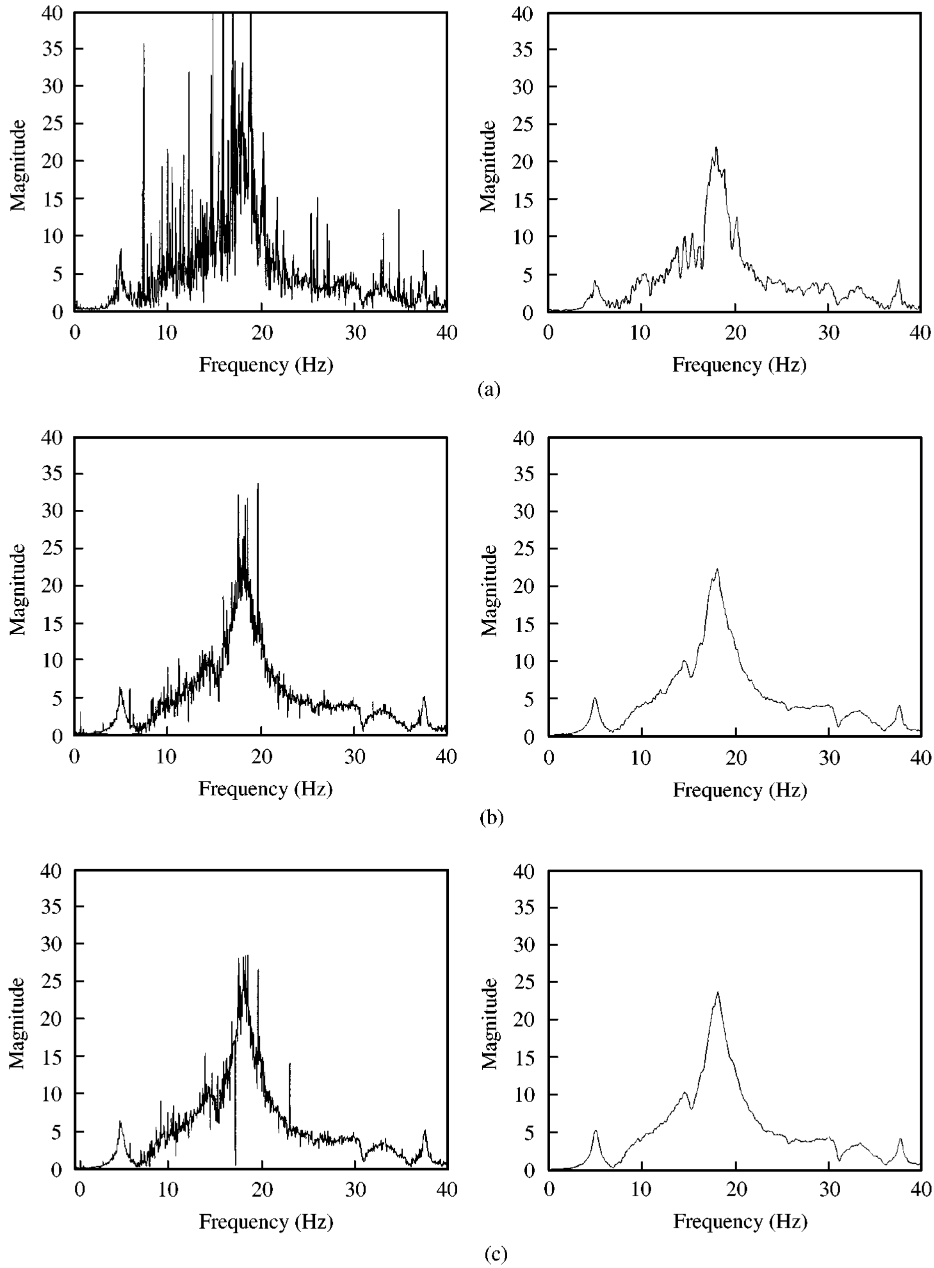

Figure 9. The estimation of the FRF magnitude by the simple mean-averaging (left) and the proposed method (right): (a) single FRF; (b) no. of averages = 20; (c) no. of averages = 40.

Here N is the number of data. It was shown that as compared with a traditional mean-averaging process, the proposed method in this work enables to shorten the existing mean-averaging process and simultaneously gives a better estimate of FRF in terms of error norm.

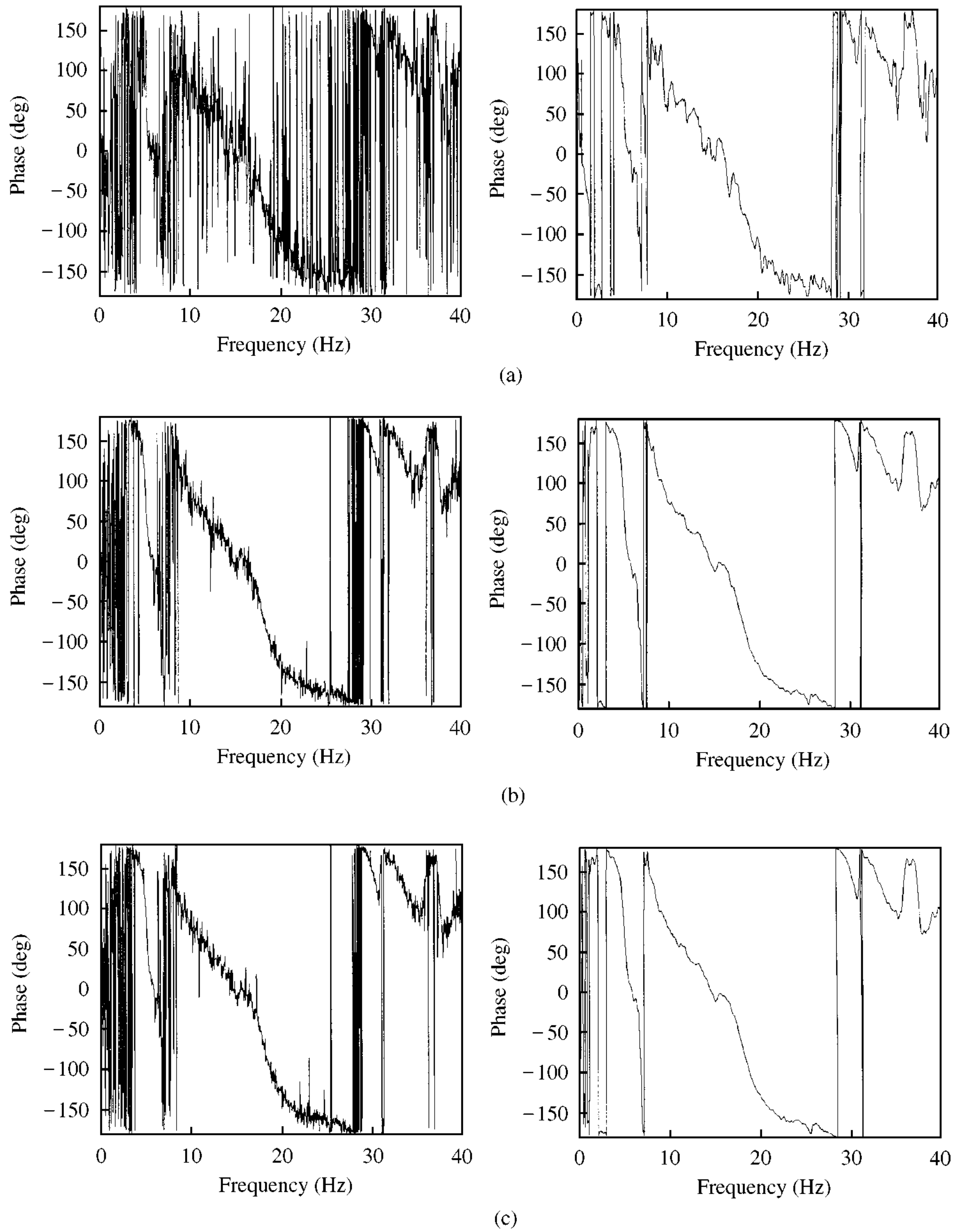

Figure 10. The estimation of the FRF phase by the simple mean-averaging (left) and the proposed method (right): (a) single FRF; (b) no. of averages = 20; (c) no. of averages = 40.

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REFERENCES

1. S. B. JULIUS and G. P. ALLAN 1986 *Random Data: Analysis and Measurement Procedures*. New York: John Wiley & Sons, Inc.; second edition.
2. D. E. NEWLAND 1993 *Random Vibration, Spectral and Wavelet Analysis*. New York: John Wiley & Sons, Inc.; third edition.
3. A. CHAMBOLLE, R. A. DEVORE, N. Y. LEE and B. J. LUCIER 1998 *IEEE Transactions on Image Processing* **7**, 391–335. Nonlinear wavelet image processing: variational problems, compression, and noise removal through wavelet shrinkage.
4. R. A. DEVORE, B. JAWERTH and B. J. LUCIER 1992 *IEEE Transactions on Information Theory* **38**, 719–746. *Image compression through wavelet transform coding*; Special issue on Wavelet Transform and Multiresolution Analysis.
5. D. L. DONOHO and I. M. JOHNSTONE 1995 *Journal of American Statistical Assoication* **90**, 1200–1244. Adapting to unknown smoothness via wavelet shrinkage.
6. D. L. DONOHO and I. M. JOHNSTONE 1994 *Biometrika* **81**, 425–455. Ideal spatial adaptation by wavelet shrinkage.
7. D. L. DONOHO 1995 *IEEE Transactions on Information Theory* **41,** 613–649. De-noising by soft-thresholding.
8. P. BODIN and B. A. WAHLBERG 1994 *Proceedings of SYSID*'94, 2441–2446. A wavelet approach to frequency response estimation.
9. P. BODIN and B. A. WAHLBERG 1998 *International Journal of Adaptive Control and Signal Processing* **12**, 407–416. A frequency response estimation method based on smoothing and thresholding.
10. A. G. BRUCE, D. L. DONOHO, H. Y. GAO and R. D. MARTIN 1994 *Proceedings of SPIE Conference* Vol. 2242, 325–336. Denoising and robust non-linear wavelet analysis.
11. I. DAUBECHIES 1992 *Ten Lectures on Wavelets*, Philadelphia: SIAM.
12. S. G. MALLAT 1998 *A Wavelet Tour of Signal Processing*, USA: Academic Press.
13. R. R. COIFMAN and D. L. DONOHO 1995 in *Translation invariant de-noising, in wavelet and statistics, Lecture Notes in Statistics* (A. Antoniadis and G. Oppenheim, editors). 125–150. Berlin Springer-Verlag.